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Conifolds with discrete torsion and noncommutativity

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Abstract: We study D3-branes at orbifolded conifold singularities in the presence of discrete torsion. The vacuum moduli space of open strings becomes non commutative due to a deformation of the superpotential and is studied via the representation theory of the moduli algebra. It is also shown that the center of the moduli algebra correctly describes the underlying orbifolded conifolds. The field theory can be obtained by a marginal deformation of the $\mathcal{N} = 1$ gauge theory on D3-branes at conifold singularity, the global symmetry being broken from $SU(2) \times SU(2)$ to $U(1) \times U(1)$. By using the AdS/CFT correspondence we argue that the marginal deformation is related to massless KK modes of NSNS and RR two form reduced on the compact space $\mathbb{T}^{1,1}$. We build a $\mathbb{T}^2$ fibration of $\mathbb{T}^{1,1}$ and show that a D3-brane in the bulk correspond to a D5-brane on the $\mathbb{T}^2$ fibre. We also discuss the possible brane construction of the system.

Keywords: D-branes, AdS/CFT Correspondance, Non-Commutative Geometry.

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1. Introduction

D3-branes probing a Calabi-Yau space provide interesting short-distance stringy geometry through its vacuum moduli space and supersymmetric gauge field theory on their world volume. If the Calabi-Yau space is smooth, then the low-energy gauge field theory is $\mathcal{N} = 4$ in four dimensions whose supergravity dual is type-IIB string theory on $AdS_5 \times S^5$ according to the AdS/CFT correspondence [1, 2]. If the Calabi-Yau space becomes singular, the story is more interesting and somewhat more natural from the point view of the AdS/CFT correspondence. The field theory we obtain this way has chiral gauge sectors which is phenomenologically more interesting and the moduli space has a rich phase structure which could include topologically distinct Calabi-Yau spaces connected by flops.

One of the most often studied case is an orbifold. The orbifold is a space obtained by taking a quotient of a smooth space X by the orbits of a discrete group G. In string theory, the discrete group G acts not only on the space X, but also on the gauge degrees of freedom which makes the moduli space smooth by producing so called twisted sectors. This idea was generalized for quotient conifolds in [6, 7, 8] and other aspects have been studied in [9]–[15].

In the orbifold theory of [16, 17], it was observed in [18] that there is an ambiguity in phases chosen for twisted sector. This can be consistently implemented in the closed string theory by weighing the path integral sector with an extra phase factor. This extra phase factor, called discrete torsion, can be classified by the second cohomology $H^2(G, \mathrm{U}(1))$ of the orbifold group G. In the open string theory, the discrete torsion was introduced in [19, 20] via projective representation of G on the Chan-Paton factors and this has been further justified and studied in [21]. Important progress has been made towards studying of orbifolds with discrete torsion in [22]–[27].

In this paper, we consider $\mathbb{Z}_k \times \mathbb{Z}_k$ orbifolds of the conifold in the presence of discrete torsion through the framework developed in [19]. In contrast to orbifold singularities, a simple conifold singularity is a singularity in the conformal field theory (in the absence of B-field) and the singularity no longer remains in the CFT if we have discrete torsion [23].

When a large number N of D3-branes are brought near the conifold singularity, the near horizon geometry is $AdS_5 \times \mathbb{T}^{1,1}$ [5] and on the world volume of D3-branes we have a chiral $\mathcal{N} = 1\, \mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory with four chiral multiplets denoted by ϕ_i and ψ_i $(i = 1, 2)$ and a quartic superpotential

$$\mathrm{Tr}(\phi_1\psi_1\phi_2\psi_2 - \phi_1\psi_2\phi_2\psi_1)\,.$$

For a $\mathbb{Z}_k \times \mathbb{Z}_l$ quotient of the conifold, the gauge symmetry of the theory becomes $\mathrm{SU}(N)^{kl} \times \mathrm{SU}(N)'^{kl}$, the gauge degrees of freedom being realized by a certain choice of the Chan-Paton matrices [6]. The matter is chiral thus the choice of Chan-Paton is extremely crucial in order to avoid the anomalies.

If we consider discrete torsion, the situation is slightly different and we discuss in detail in section 3. Our result is similar to the orbifold case treated recently in [27, 28, 29].

The models with discrete torsion are interesting from the viewpoint of noncommutative geometry and in this case the noncommutativity is not in the field theory itself but is realized on the moduli space of the gauge theory. In terms of branes, the space *orthogonal* to the branes becomes non commutative. The noncommutativity in our case is related to a q-deformed Heisenberg algebra. Because the moduli represents a solution of both F-term and D-term equations of field theory, in the representations of the non-commutative algebra only products $\phi\psi$ appear because of D-term equations.

We will also discuss massless states of the 5-dimensional supergravity by looking at the eigenvalues of the laplacian on $\mathbb{T}^{1,1}$ compact space. In the field theory side, the superpotential is a marginal deformation which preserves a $\mathrm{U}(1) \times \mathrm{U}(1)$ global symmetry. In the supergravity side, a marginal deformation corresponds to massless KK modes and we give an argument that these massless KK modes come from reducing the RR and NS forms from 10 dimensions on the compact space. This will determine a deformation of the compact space on a $\mathbb{T}^2$ fibration of $\mathbb{T}^{1,1}$ and the global symmetry $\mathrm{U}(1) \times \mathrm{U}(1)$ will be identified with the symme-

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try on a 2-torus appearing as a fibration of $\mathbb{T}^{1,1}$. The identification becomes very natural in the conifold case and we can also connect our results to those without discrete torsion.

The paper is organized as follows. In section 2 we give a detailed mathematical description of the projective representation. In section 3 we present a complete discussion of the field theory on D3-branes orthogonal to orbifolded conifolds with discrete torsion. In section 4 we make comments on the massless KK modes which appear within the AdS/CFT duality and correspond to the marginal deformation of the superpotential. In section 5 we speculate on the realization of some aspects of orbifolded conifolds with discrete torsion with brane construction.

2. Projective representation and discrete torsion

In [16, 17], string theory on an orbifold X/G is obtained by projecting out G non-invariant subspace from Hilbert space of strings on X where strings are allowed to be closed up to the action of G. In [18], it was discovered that due to twisting from G non-trivial phase factors appear in the one-loop partition function of closed string orbifold theory and they are consistent with modular invariance if these form a discrete torsion, i.e. an element of $H^2(G, \mathrm{U}(1))$. When D-branes are introduced in this picture the low energy effective field theory is constructed using quiver technique with projective representations of the orbifold group [19, 20, 21]. Below we clarify some technical aspects related to projective representation and discrete torsion.

A mapping $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ is called a projective representation of G if there exists a mapping $\alpha : G \times G \to \mathrm{U}(1)$ such that

$$\rho(g)\rho(h) = \alpha(g,h)\rho(gh)\,, \qquad \rho(e) = I_n\,, \tag{2.1}$$

for all elements $g, h \in G$, where e denotes the identity element of Γ and I_n denotes the $n \times n$ identity matrix. The mapping α is called the factor system of the projective representation γ. From the associativity of G, we obtain

$$\alpha(g,e) = \alpha(e,g) = 1\,, \qquad \alpha(g,h)\alpha(gh,k) = \alpha(g,hk)\alpha(h,k) \quad \forall\, g,h,k \in G\,. \tag{2.2}$$

The mapping α satisfying these properties is called a cocyle. The mapping α is called a coboundary if there is a mapping $\gamma : G \to \mathrm{U}(1)$ such that $\alpha(g,h) = \gamma(g)\gamma(h)\gamma^{-1}(gh)$. Two cocyles are equivalent if their quotient is a coboundary. The set of equivalence classes of cocycles form a group under multiplication which will be denoted by $H^2(G, \mathrm{U}(1))$. A projective representation gives rise to a central extension E of G by $\mathrm{U}(1)$ and a representation $\tilde{\rho} : E \to U(n, \mathbb{C})$ so that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{U}(1) \arrow[r] \arrow[d, "\mathrm{id}"'] & E \arrow[r] \arrow[d, "\tilde{\rho}"'] & G \arrow[r] \arrow[d, "\hat{\rho}"'] & 1 \\
1 \arrow[r] & \mathrm{U}(1) \arrow[r] & \mathrm{GL}(n,\mathbb{C}) \arrow[r] & \mathrm{PGL}(n,\mathbb{C}) \arrow[r] & 1
\end{tikzcd}$$

where $\mathrm{PGL}(n,\mathbb{C})$ is the quotient of $\mathrm{GL}(n,\mathbb{C})$ by $\mathrm{U}(1)$ and the map $\hat{\rho} : G \to \mathrm{PGL}(n,\mathbb{C})$ is the composition of ρ with the natural projection $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$. It can be shown that there is one-to-one correspondence between the central extensions of G by $\mathrm{U}(1)$ and the projective representations of G. Two projective representations ρ_1, ρ_2 are said to be projectively equivalent if $\hat{\rho}_1 = \hat{\rho}_2$. Thus the projective representations are up to equivalences classified by $\mathrm{Ext}(G, \mathrm{U}(1))$ which is equal to $H^2(G, \mathrm{U}(1))$. Given a cocycle α, one can define a twisted group algebra

$$\mathbb{C}_\alpha G = \left\{ \sum c_i \bar{g}_i | c_i \in \mathbb{C}, \ g_i \in G \right\}, \tag{2.3}$$

with multiplication

$$\bar{g}\bar{h} = \alpha(g,h)\overline{gh} \quad \forall\, g,h \in G\,. \tag{2.4}$$

Then there is one-to-one correspondence between α-representations of G and $\mathbb{C}_\alpha G$-modules. For a given α-representation ρ on the vector space V, V becomes an $\mathbb{C}_\alpha G$-module via a homomorphism:

$$h : \mathbb{C}_\alpha G \longrightarrow \mathrm{End}_{\mathbb{C}}(V)\,. \tag{2.5}$$

Hence we identify α-representations with $\mathbb{C}_\alpha G$-modules.

In this paper, we are interested in the case $G = \mathbb{Z}_k \times \mathbb{Z}_l$. It can be shown that [30]

$$H^2(\mathbb{Z}_k \times \mathbb{Z}_l, \mathrm{U}(1)) = \mathrm{Hom}(\mathbb{Z}_k \otimes \mathbb{Z}_l, \mathrm{U}(1)) = \mathbb{Z}_p\,, \tag{2.6}$$

where $p = \gcd(k,l)$.

The phase $\beta(g,h)$ appearing in the closed string theory in the (g,h)-twisted sector is of the form

$$\beta(g,h) = \alpha(g,h)\alpha(h,g)^{-1}\,, \tag{2.7}$$

since G is abelian. Then β depends only on the equivalence classes of the α and satisfies the following cocyle condition

$$\begin{gathered}
\beta(g,g) = 1\,, \qquad \beta(g,h) = \beta(h,g)^{-1}\,, \\
\beta(g,hk) = \beta(g,h)\beta(g,k)\,, \quad \forall\, g,h,k \in G\,.
\end{gathered} \tag{2.8}$$

These properties completely fix the β cocycles. Indeed, the elements of $H^2(\mathbb{Z}_k \times \mathbb{Z}_l, \mathrm{U}(1))$ are of the form

$$\beta((a,b),(a',b')) = \omega_p^{m(ab'-a'b)}\,, \qquad \omega_p = e^{2\pi i/p}\,,\ m = 1,\ldots,p\,, \tag{2.9}$$

where $(a,b),(a',b') \in \mathbb{Z}_k \times \mathbb{Z}_l$. The different projective representations are therefore determined by the parameter $\epsilon = \omega_p^m$ and we have

$$\rho(a,b)\rho(a',b') = \epsilon^{-ba'}\rho(a+a',b+b')\,. \tag{2.10}$$

Let s be the smallest non-zero number such that $\epsilon^s = 1$. Any irreducible α-representation is projectively equivalent to

$$\rho(a,b) = P^a Q^b\,, \tag{2.11}$$

where

$$P = \mathrm{diag}(1,\epsilon^{-1},\epsilon^{-2},\ldots,\epsilon^{-(s-1)})\,, \qquad Q = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.12}$$

The number of irreducible projective representations of G with cocycle α equals the number of α-regular elements of G. An element $g \in G$, for abelian G, is α-regular if

$$\alpha(g,h) = \alpha(h,g)\,, \quad \forall\, h \in G\,. \tag{2.13}$$

Thus the number N_α of irreducible projective representations with cocycle class α is given by

$$N_\alpha = \frac{1}{|G|}\sum_{g,h}\frac{\alpha(g,h)}{\alpha(h,g)} = \frac{1}{|G|}\sum_{g,h}\beta(g,h)\,. \tag{2.14}$$

All the (linearly different) irreducible α-representations $R^{\mathrm{irr}}_{i,j}$ can be obtained by multiplying by phases:

$$\rho_{i,j}(1,0) = \omega_k^i\rho(1,0)\,, \qquad \rho_{i,j}(0,1) = \omega_l^j\rho(0,1)\,, \tag{2.15}$$

where $i = 0,\ldots,k/s-1$, and $j = 0,\ldots,l/s-1$. Since $\mathbb{C}_\alpha G$-modules are completely reducible, a general projective representation R is a direct sum of irreducible representations $R^{\mathrm{irr}}_{i,j}$:

$$R = \oplus m_{i,j}R^{\mathrm{irr}}_{i,j}\,. \tag{2.16}$$

The twisted group algebra $\mathbb{C}_\alpha G$ itself can be regarded as a $\mathbb{C}_\alpha G$ module. The α-representation corresponding to $\mathbb{C}_\alpha G$ is called the regular α-representation of G. When $G = \mathbb{Z}_k \times \mathbb{Z}_k$, the regular representation $\mathbb{C}_\alpha G$ is equal to kR^{irr}.

3. The orbifolded conifolds and the gauge theory of the branes

Now we consider quotient singularities of the conifold (i.e. orbifolded conifold). The conifold is a three dimensional hypersurface singularity in $\mathbb{C}^4$ defined by:

$$\mathcal{C}:\ xy - uv = 0\,. \tag{3.1}$$

The conifold can be realized as a holomorphic quotient of $\mathbb{C}^4$ by the $\mathbb{C}^*$ action given in [5]

$$(A_1, A_2, B_1, B_2) \longmapsto (\lambda A_1, \lambda A_2, \lambda^{-1} B_1, \lambda^{-1} B_2) \quad \text{for } \lambda \in \mathbb{C}^* . \tag{3.2}$$

Thus the charge matrix is the transpose of $Q' = (1, 1, -1, -1)$ and $\Delta = \sigma$ is a convex polyhedral cone in $\mathbb{N}'_{\mathbf{R}} = \mathbf{R}^3$ generated by $v_1, v_2, v_3, v_4 \in \mathbf{N}' = \mathbb{Z}^3$ where

$$v_1 = (1, 0, 0)\,, \qquad v_2 = (0, 1, 0)\,, \qquad v_3 = (0, 0, 1)\,, \qquad v_4 = (1, 1, -1)\,. \tag{3.3}$$

The isomorphism between the conifold $\mathcal{C}$ and the holomorphic quotient is given by

$$x = A_1 B_1\,, \qquad y = A_2 B_2\,, \qquad u = A_1 B_2\,, \qquad v = A_2 B_1\,. \tag{3.4}$$

We take a further quotient of the conifold $\mathcal{C}$ by a discrete group $\mathbb{Z}_k \times \mathbb{Z}_l$. Here $\mathbb{Z}_k$ acts on A_i, B_j as

$$(A_1, A_2, B_1, B_2) \longmapsto (e^{2\pi i/k} A_1, A_2, e^{-2\pi i/k} B_1, B_2)\,, \tag{3.5}$$

and $\mathbb{Z}_l$ acts as

$$(A_1, A_2, B_1, B_2) \longmapsto (e^{2\pi i/l} A_1, A_2, B_1, e^{-2\pi i/l} B_2)\,. \tag{3.6}$$

Thus they will act on the conifold $\mathcal{C}$ as

$$(x, y, u, v) \longmapsto (x, y, e^{2\pi i/k} u, e^{-2\pi i/k} v)\,, \tag{3.7}$$

and

$$(x, y, u, v) \longmapsto (e^{2\pi i/l} x, e^{-2\pi i/l} y, u, v)\,. \tag{3.8}$$

This quotient is called the orbifolded conifold or the hyper-quotient of the conifold and denoted by $\mathcal{C}_{kl}$.

Consider a system of M D3-branes sitting in the transversal direction of the orbifolded conifold in $\mathbb{R}^{1,3} \times \mathcal{C}_{kk}$. The corresponding supersymmetric gauge field theory on the world volume of the D3-branes for the case of the conifold was constructed by Klebanov and Witten [5] guided by the toric description of the conifold as explained above. The parameters A_i and B_j give rise to the chiral superfields transforming as $(\square, \bar{\square})$ and $(\bar{\square}, \square)$, respectively, with respect to the gauge group $\mathrm{SU}(M) \times \mathrm{SU}(M)$. There is also an additional anomaly-free U(1) R-symmetry, under which A_i and B_j both have charge 1/2.

In our case, we have orbifolded conifolds with discrete torsion $\epsilon = e^{2\pi i/k} \in H^2(\mathbb{Z}_k \times \mathbb{Z}_k, \mathrm{U}(1))$. As in [6], we begin with $\mathrm{SU}(k^2 M) \times \mathrm{SU}(k^2 M)$ gauge theory with the Chan-Paton degrees of freedom corresponding to the regular representation;

$$R(a, b) = P^a Q^b \otimes 1_{kM}\,. \tag{3.9}$$

This breaks the gauge group to $\mathrm{SU}(kM)^k \times \mathrm{SU}(kM)^k$ and the gauge field projects

$$R(a,b) G_\mu R(a,b)^{-1} = G_\mu \,, \tag{3.10}$$

and the chiral superfields project according to

$$R(a,b) A_i R(a,b)^{-1} = (a,b) \cdot A_i \,, \qquad R(a,b) B_i R(a,b)^{-1} = (a,b) \cdot B_i \,. \tag{3.11}$$

For the irreducible representation, the solution is of the form

$$A_1 = P^{-1} Q \,, \qquad A_2 = I \,, \qquad B_1 = Q^{-1} \,, \qquad B_2 = P \,. \tag{3.12}$$

Hence, for the regular representation, the most general solution is obtained by tensoring this $k \times k$ solution with $N \times N$ matrices. After tensoring (3.12) with $N \times N$ matrices and substituting into the $\mathrm{SU}(kN) \times \mathrm{SU}(kN)$ theory, we obtain an $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(N) \times \mathrm{SU}(N)'$ gauge theory. The matter content is in the following representations:

Fields	**Representations**
(ϕ_1)	$(\ \square\,,\ \bar{\square}\,'\,)$
(ϕ_2)	$(\ \square\,,\ \bar{\square}\,'\,)$
(ψ_1)	$(\ \bar{\square}\,,\ \square\,'\,)$
(ψ_2)	$(\ \bar{\square}\,,\ \square\,'\,)$

The superpotential is

$$\begin{aligned} W &\sim \mathrm{Tr}\,\big((P^{-1}Q \otimes \phi_1)(Q^{-1} \otimes \psi_1)(I \otimes \phi_2)(P \otimes \psi_2) - \\ &\quad - (P^{-1}Q \otimes \phi_1)(P \otimes \psi_2)(I \otimes \phi_2)(Q^{-1} \otimes \psi_1)\big) \\ &\sim \mathrm{Tr}(\phi_1 \psi_1 \phi_2 \psi_2 - \epsilon^{-1} \phi_1 \psi_2 \phi_2 \psi_1) \,. \end{aligned} \tag{3.13}$$

Before going any further let us comment on the global symmetries which are preserved by this superpotential. We start with a short discussion on some marginal deformation of the $\mathcal{N} = 4$ theory. Written in an $\mathcal{N} = 1$ notation, the interactions are summarized in the superpotential:

$$W \sim \mathrm{Tr}([\Phi_1, \Phi_2] \Phi_3) \,, \tag{3.14}$$

where Φ_i, $i = 1, 2, 3$ are the chiral multiplets components of the $\mathcal{N} = 4$ multiplet. In this notation, only the $\mathrm{U}(1)_R$ symmetry of the $\mathcal{N} = 1$ supersymmetry and an $\mathrm{SU}(3)$ that rotates the Φ_i fields are visible. In [19, 20, 27, 28, 29] the orbifold with discrete torsion has been discussed and the global symmetry group becomes different. The global symmetry $\mathrm{SU}(3)$ breaks into its Cartan subalgebra $\mathrm{U}(1) \times \mathrm{U}(1)$ (the charges of one of the three fields are determined by the charges of the other two fields) and this together with the $\mathrm{U}(1)_R$ symmetry will determine a $\mathrm{U}(1)^3$ symmetry with different phases for all the three fields.

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In our case, we compare with the theory obtained on D3-branes at a conifold singularity [5]. The superpotential preserves a $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_R$ symmetry, the first SU(2) acting on the A_i fields, the second SU(2) acts on the B_i fields and all the A_i, B_i fields have an R-symmetry charge equal to 1/2. In the case of orbifolded conifolds with discrete torsion, the global symmetry breaks to its Cartan subalgebra $\mathrm{U}(1) \times \mathrm{U}(1)$, where the first U(1) acts on ϕ_1, ϕ_2 by $e^{i\theta}\phi_1, e^{-i\theta}\phi_2$ and the second U(1) acts on ψ_1, ψ_2 by $e^{i\theta}\psi_1, e^{-i\theta}\psi_2$. Therefore the global symmetry of the superpotential (3.13) is $\mathrm{U}(1)^2 \times \mathrm{U}(1)_R$.

The superpotential has conformal dimension 3 and R charge 2 so it is a marginal deformation. By using the AdS/CFT correspondence [1, 2], it should correspond to a massless KK mode in supergravity, and we will discuss this issue in section 4.

After we discussed the symmetries we can go further to describe the equations derived from it. The F-term equation for the vacuum will be

$$\begin{aligned} \psi_1\phi_2\psi_2 - \epsilon^{-1}\psi_2\phi_2\psi_1 &= 0\,, \qquad & \psi_2\phi_1\psi_1 - \epsilon^{-1}\psi_1\phi_1\psi_2 &= 0\,, \\ \phi_2\psi_2\phi_1 - \epsilon^{-1}\phi_1\psi_2\phi_2 &= 0\,, \qquad & \phi_1\psi_1\phi_2 - \epsilon^{-1}\phi_2\psi_1\phi_1 &= 0\,. \end{aligned} \tag{3.15}$$

These relations indicate the moduli space is non-commutative once we introduce the discrete torsion ϵ.

Note that, besides the F-term equation, we also have the D-flatness condition which implies that

$$|\phi_1|^2 + |\phi_2|^2 - |\psi_1|^2 - |\psi_2|^2 = \zeta\,. \tag{3.16}$$

In the next section, we present the ideas of describing the moduli space following [28].

3.1 Non-commutative moduli space

In [28], the authors proposed that one needs to express the moduli space of vacua in terms of non-commutative algebraic geometry to capture D-brane physics correctly. A general framework has been developed in [31, 32].

Locally the moduli space can be described in terms of finitely generated associative algebras over $\mathbb{C}$ with unity. Globally, we need to glue together the locally ringed (non-commutative) spaces constructed below. For a given moduli algebra $\mathcal{A}$, let $\mathcal{ZA}$ be the center of $\mathcal{A}$. Then $\mathcal{ZA}$ is a commutative algebra and we may associate a geometric object whose points consist of prime ideals in $\mathcal{ZA}$:

$$\mathrm{Spec}\mathcal{ZA} = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal in } \mathcal{ZA}\}\,. \tag{3.17}$$

We can endow $\mathrm{Spec}\mathcal{ZA}$ with a natural topology where the smallest closed set containing a prime ideal $\mathfrak{p}$ consists of all prime ideals containing $\mathfrak{p}$. This topology is called Zariski topology on $\mathrm{Spec}\mathcal{ZA}$.

Now we assume that $\mathcal{A}$ is a finite $\mathcal{ZA}$-module. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{ZA}$ which is the set of all functions vanishing at a closed point in $\mathcal{ZA}$. Then $\mathfrak{m}\mathcal{A}$ will be two sided proper ideal in $\mathcal{A}$ since $\mathcal{A}$ is a finite $\mathcal{ZA}$-module. Moreover we have an injective homomorphism

$$\frac{\mathcal{ZA}}{\mathfrak{m}} \longrightarrow \frac{\mathcal{A}}{\mathfrak{m}\mathcal{A}} . \tag{3.18}$$

Therefore $\mathcal{A}/\mathfrak{m}\mathcal{A}$ is an algebra over $\mathbb{C}$ which is finite dimensional as a vector space by our assumption on finiteness. Then we look for a map into an algebra of $M \times M$ matrices over $\mathbb{C}$

$$\pi : \frac{\mathcal{A}}{\mathfrak{m}\mathcal{A}} \longrightarrow \mathrm{Mat}(M, \mathbb{C}) , \tag{3.19}$$

whose image will give an irreducible representation in a sense that $\mathbb{C}^M$ will be the only non-trivial space which is invariant under $\pi(\mathcal{A}/\mathfrak{m}\mathcal{A})$. Note that the ideal $\mathfrak{m}\mathcal{A}$ will not be a maximal ideal of $\mathcal{A}$. Otherwise, the algebra $\mathcal{A}/\mathfrak{m}\mathcal{A}$ will be an algebraic division algebra over $\mathcal{ZA}/\mathfrak{m} \cong \mathbb{C}$. Thus $\mathcal{A}/\mathfrak{m}\mathcal{A}$ will be isomorphic to $\mathbb{C}$ and all the representations will be one dimensional, though this is not the case in general. We will investigate all possible such maps and their images. We will find that there is a unique map up to $\mathrm{GL}(M, \mathbb{C})$ conjugate action on $\mathrm{Mat}(M, \mathbb{C})$ at a generic point of $\mathcal{ZA}$, but there could be many different irreducible representations at special points. The representations will be parameterized by the space Spec $\mathcal{ZA}$ which will be irreducible at generic points, but as we approach to the singular points of the orbifolded conifolds, the representation breaks into a direct sum of irreducible representations. Hence we may form fractional branes. This leads to consider the symmetric spaces of the moduli which is a free abelian group generated by all possible irreducible representations arising in this manner. The symmetric space was denoted by $\mathcal{SM}_{\mathcal{A}}$ in [28].

3.2 The moduli space of the orbifolded conifolds with discrete torsion

In our case, eqs. (3.15) show that the vacuum moduli $\mathcal{M}$ is non-commutative. Let $\mathcal{M}_F$ be the vacuum moduli with only F-term constraints. Then the corresponding moduli algebra $\mathcal{A}_F$ is generated by ϕ_i, ψ_i's. But to get gauge invariant moduli, we also have to impose the D-term equation. The D-term equation is given by a $\mathbb{C}^*$ action on ϕ_i, ψ_i, i.e.

$$\lambda \cdot (\phi_1, \phi_2, \psi_1, \psi_2) = (\lambda\phi_1, \lambda\phi_2, \lambda^{-1}\psi_1, \lambda^{-1}\psi_2) , \quad \text{for } \lambda \in \mathbb{C}^* . \tag{3.20}$$

Hence the final moduli algebra $\mathcal{A}$ is generated by $\mathbb{C}^*$ invariant fields $\phi_1\psi_1$, $\psi_1\phi_1$, $\phi_1\psi_2$, $\psi_2\phi_1$, $\phi_2\psi_1$, $\psi_1\phi_2$, $\phi_2\psi_2$, $\psi_2\phi_2$ with constraints (3.15).

To reduce the number of the generators of $\mathcal{A}$ by half, we now carefully compare the U(1) R-symmetries of the conifold and the field theory as in [5]. First note that the transformation which multiplies each coordinate by $e^{i\theta}$ acts on the canonical bundle K by multiplication by $e^{2i\theta}$. Hence it acts on the chiral superspace

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(which transforms as $\sqrt{K}$) coordinates by $e^{i\theta}$. Consider $\theta = 2\pi/k$. This gives an element of the R-symmetry group that acts on the conifold by ϵ and on the chiral superspace by ϵ. On the gauge theory side, this transformation corresponds to the action $A_i \to e^{\pi i/k} A_i$ and $B_j \to e^{\pi i/k}$ since A_i and B_j have R-charge 1/2. From the toric description, the exchange of A_i and B_j is $\mathbb{Z}_2$ discrete symmetry of the conifold. In the field theory side, the exchange of A_i and B_j will change the superpotential $W = \mathrm{Tr}\,(\phi_1\psi_1\phi_2\psi_2 - \epsilon^{-1}\phi_1\psi_2\phi_2\psi_1)$. To compensate this change in the superpotential, the R-symmetry Υ was introduced in [5] which acts on chiral superspace coordinates by $\theta \to i\theta$, acts on gluinos by $\lambda \to i\lambda$, leaves invariant the superfields A and B, and therefore acts on femionic components $\mathcal{F}$ of A or B by $\mathcal{F} \to -i\mathcal{F}$. Hence we may assume that $\mathcal{A}$ is generated by

$$\phi_1\psi_1\,, \qquad \phi_1\psi_2\,, \qquad \phi_2\psi_1\,, \qquad \phi_2\psi_2\,. \tag{3.21}$$

after combining the exchange of A_i and B_j and Υ transformation. But these generators are not independent and they satisfy

$$\phi_1\psi_1\phi_2\psi_2 = \epsilon^{-1}\phi_1\psi_2\phi_2\psi_1\,. \tag{3.22}$$

We will denote the monomial

$$(\phi_1\psi_1)^{a_1}(\phi_1\psi_2)^{a_2}(\phi_2\psi_1)^{a_3}(\phi_2\psi_2)^{a_4}\,, \tag{3.23}$$

by (a_1, a_2, a_3, a_4) and the multiplication

$$\left[(\phi_1\psi_1)^{b_1}(\phi_1\psi_2)^{b_2}(\phi_2\psi_1)^{b_3}(\phi_2\psi_2)^{b_4}\right] \cdot \left[(\phi_1\psi_1)^{a_1}(\phi_1\psi_2)^{a_2}(\phi_2\psi_1)^{a_3}(\phi_2\psi_2)^{a_4}\right], \tag{3.24}$$

by $(b_1, b_2, b_3, b_4)\cdot(a_1, a_2, a_3, a_4)$. Let us first study the commutative part of the moduli algebra $\mathcal{A}$. The monomials

$$(a_1, a_2, a_3, a_4) \tag{3.25}$$

will be in the center $\mathcal{ZA}$ if and only if

$$(b_1, b_2, b_3, b_4)\cdot(a_1, a_2, a_3, a_4) = (a_1, a_2, a_3, a_4)\cdot(b_1, b_2, b_3, b_4) \tag{3.26}$$

for every monomial (b_1, b_2, b_3, b_4). Using the relation (3.15), we obtain

$$\begin{aligned}(b_1, b_2, b_3, b_4)\cdot(a_1, a_2, a_3, a_4) &= \epsilon^{(b_2-b_3)(a_4-a_1)-(b_4-b_1)(a_2-a_3)} \times \\ &\quad \times (a_1, a_2, a_3, a_4)\cdot(b_1, b_2, b_3, b_4)\,.\end{aligned} \tag{3.27}$$

Thus the monomial $(\phi_1\psi_1)^{a_1}(\phi_1\psi_2)^{a_2}(\phi_2\psi_1)^{a_3}(\phi_2\psi_2)^{a_4}$ is in the center iff

$$a_1 = a_4, a_2 = a_3 \pmod{k}\,. \tag{3.28}$$

Let

$$\begin{aligned} x &= (\phi_1\psi_1)^k\,, \qquad y = (\phi_2\psi_2)^k\,, \qquad u = (\phi_1\psi_2)^k\,, \\ v &= (\phi_2\psi_1)^k\,, \qquad z = \phi_1\psi_1\phi_2\psi_2\,. \end{aligned} \tag{3.29}$$

From the relations (3.22) and (3.28), we can see that the center $\mathcal{ZA}$ can be expressed as

$$xy = z^k \,, \qquad uv = z^k \,, \tag{3.30}$$

which is exactly the orbifolded conifold. Therefore we see that the orbifolded conifold space is described by the commutative part of the algebra.

After identifying the commutative center of the moduli algebra we now consider the non-commutative points of the moduli. Let

$$(x - x_0, y - y_0, u - u_0, v - v_0, z - z_0)\mathcal{ZA} \,, \tag{3.31}$$

be the maximal ideal corresponding to a point $x = x_0, y = y_0, u = u_0, v = v_0, z = z_0$ on the orbifolded conifold, where $z_0^k = x_0 y_0 = u_0 v_0$. The corresponding non-commutative points will be given by

$$\frac{\mathcal{A}}{((\phi_1\psi_1)^k - x_0, (\phi_2\psi_2)^k - y_0, (\phi_1\psi_2)^k - u_0, (\phi_2\psi_1)^k - v_0, \phi_1\psi_1\phi_2\psi_2 - z_0)\,\mathcal{A}} \,. \tag{3.32}$$

Now we look for irreducible representations π of this algebra, that is, a map into a matrix algebra whose image forms an irreducible representation.

First we consider the most generic point $(x_0, y_0, u_0, v_0, z_0)$ on Spec $\mathcal{A}$, by which we mean that none of x_0, u_0, v_0, y_0 are zero. Then the minimal polynomials of $\pi(\phi_1\psi_1)$ and $\pi(\phi_2\psi_2)$ will divide

$$\pi^k(\phi_1\psi_1) - x_0 = 0 \,, \qquad \pi^k(\phi_2\psi_2) - y_0 = 0 \,, \tag{3.33}$$

respectively. Moreover, $\pi(\phi_1\psi_1)$ and $\pi(\phi_2\psi_2)$ commutes. Hence there must be a common eigenvector with eigenvalues a and d, respectively where $a^k = x_0$, $d^k = y_0$. We denote the common eigenvector with these eigenvalues by $|[a, 0, 0, d]_0>$ By acting $\pi^i(\phi_1\psi_2)$, $i = 1, \ldots, k-1$ on $|[a, 0, 0, d]_0>$, we obtain a collection of vectors

$$\begin{aligned} &|[a,0,0,d]_0>, |[a,0,0,d]_1>, \ldots, |[a,0,0,d]_{k-1}> \,, \\ &\text{with } |[a,0,0,d]_i> \equiv \pi^i(\phi_1\psi_2)|[a,0\,0,d]_0> \,, \end{aligned} \tag{3.34}$$

which are simultaneously eigenvectors of $\pi(\phi_1\psi_1)$ with eigenvalues $a, \epsilon a, \ldots, \epsilon^{(k-1)}\, a$ and eigenvectors of $\pi(\phi_2\psi_2)$ with eigenvalues $d, \epsilon^{-1}\, d, \ldots, \epsilon^{-(k-1)}\, d$. A set of matrices which satisfies these conditions is

$$\pi(\phi_1\psi_1) = aP^{-1} \,, \qquad \pi(\phi_1\psi_2) = bQ^{-1} \,, \qquad \pi(\phi_2\psi_2) = dP \,, \tag{3.35}$$

where P, Q are defined in (2.12). It also follows from (3.22) that $\pi(\phi_2\psi_1) = cQ$. From this construction, it is clear that it gives rise to an irreducible representation of rank k and we denote this by $R(a, b, c, d)$. It is also clear that this is the only possible irreducible representation up to $\mathrm{GL}(k,\mathbb{C})$ conjugate action, i.e. up to change of the basis of $\mathbb{C}^k$. We remark that

$$\begin{aligned}
&\pi(x) = a^k I\,, \qquad \pi(y) = d^k I\,, \qquad \pi(u) = b^k I\,,\\
&\pi(v) = c^k I\,, \qquad \pi(z) = adI = \omega bcI\,,
\end{aligned} \tag{3.36}$$

where ω is the k-th root of the unity and these solutions parameterize the orbifolded conifold generically, i.e. none of the coordinates are zero.

Second we suppose x_0 and u_0 are not zero, but $y_0 = v_0 = 0$. Then z_0 will be also zero and we have $\pi(\phi_1\psi_1)$ and $\pi(\phi_1\psi_2)$ satisfy $\pi^k(\phi_1\psi_1) - x_0 = 0$ and $\pi^k(\phi_1\psi_2) - u_0 = 0$ and are invertible. Thus we may find an eigenvector $|a>$ of $\pi(\phi_1\psi_1)$ with eigenvalue a where $a^k = x_0$. By acting $\pi(\phi_1\psi_2)$ on $|a>$ repeatedly, we obtain a set of eigenvectors

$$|a>, |\epsilon a>, \ldots, |\epsilon^{k-1}a>\,, \qquad \text{where } |\epsilon^i a> \equiv \pi^i(\phi_1\psi_2) \tag{3.37}$$

of $\pi(\phi_1\psi_1)$ with eigenvalues $a, \epsilon a, \ldots, \epsilon^{k-1}a$. On the other hand, $\pi(\phi_2\psi_2) = \pi(\phi_2\psi_1) = 0$ since $\pi(\phi_1\psi_1)\pi(\phi_2\psi_2) = \epsilon^{-1}\pi(\phi_1\psi_2)\pi(\phi_2\psi_1) = 0$ and $\pi(\phi_1\psi_1)$, $\pi(\phi_1\psi_2)$ are invertible, hence the representation $R(a,b,c.d)$ remains to be irreducible, i.e.

$$\lim_{u_0,v_0\to 0} R(a,b,c,d) = R(a,b,0,0)\,, \tag{3.38}$$

as expected from the fact that $c=0, d=0, e=0$ is a smooth point. The other cases with only two coordinates zero are similar.

Now it remains to consider the singular points away from the vertex of the orbifolded conifold , for example, $y_0 = u_0 = v_0 = z_0 = 0$, but $x_0 \neq 0$. Then $\pi(\phi_2\psi_2)$ is zero because $\pi(\phi_1\psi_1)$ is invertible and $\pi(\phi_1\psi_1)\pi(\phi_2\psi_2) = 0$. It is also easy to see that $\pi(\phi_2\psi_1) = \pi(\phi_1\psi_2) = 0$ by applying $\pi(\phi_2\psi_1)$ and $\pi(\phi_1\psi_2)$ repeatedly to an eigenvector of $\pi(\phi_1\psi_1)$. Since $\pi(\phi_1\psi_1)$ satisfies the equation $\pi^k(\phi_1\psi_1) - x_0 = 0$ and other operators are zero, $\pi(\phi_1\psi_1)$ decomposes into one-dimensional representations with chracters $a, \epsilon a, \ldots, \epsilon^{k-1}a$ with $a^k = x_0$, that is,

$$\lim_{y_0,u_0,v_0\to 0,} R(a,b,c,d) = R(a,0,0,0) \oplus R(\epsilon a,0,0,0) \oplus R(\epsilon^{n-1}a,0,0,0)\,. \tag{3.39}$$

Hence we can fractionalize the branes along the singular locus which contributes new twisted states along the fixed locus of the orbifolding action $\mathbb{Z}_k \times \mathbb{Z}_k$ on the conifold.

4. The AdS/CFT correspondence

In this section we will discuss issues concerning the near-horizon limit for D3-branes at orbifolded conifolds with discrete torsion singularities.

In [27, 28, 29] a comparison was made between the field theory marginal and relevant deformations and the corresponding deformations of $AdS_5 \times S^5$ for a maximal supersymmetric theory. The marginal deformations are related to massless states in the 5-dimensional supergravity [33, 34], i.e. to the vevs for harmonics of RR and NSNS fields. In the orbifold case, the presence of NS and RR fields determines a non-commutative moduli space for the D-branes and the $1-\epsilon^{-1}$ deformation correspond to background values for the RR 3-form. In the presence of the RR background field the D3-branes pick up a dipole moment for higher brane charge and become extended in two additional directions [35, 36]. In the orbifold with discrete torsion case, a D3-brane in the bulk becomes a D5-brane wrapped on a 2-torus which is a fibration of the five sphere.

In our case, we will observe a complex massless scalar field obtained by reducing the two forms of type IIB on $\mathbb{T}^{1,1}$ by using results of [37]–[42] and this corresponds to the marginal deformation of the field theory considered in the previous section. Deformations which do not preserve conformality were described in [43]–[46].

We begin the search for the massless Kaluza-Klein with right properties such that it could correspond to the marginal deformation discussed in section 3. In the absence of a consistent 5D theory (by compactification on $\mathbb{T}^{1,1}$) it is a very difficult problem to pin point the exact harmonic. Nevertheless, it is suggestive that the harmonic should be related to the complex scalar which descends from the NSNS and RR two forms whose components are all inside $\mathbb{T}^{1,1}$. To study the AdS dual of such case and similar other cases we list some CFT operators and their corresponding AdS dual [37, 39, 40, 41]:

Operators	Δ^k	r	**Multiplet**	E_0	j, l
$\mathrm{Tr}(\phi\psi)^k$	$\frac{3}{2}k$	k	vector	$\frac{3}{2}k$	$\frac{k}{2}$
$\mathrm{Tr}(W_\alpha(\phi\psi)^k)$	$\frac{3}{2}k+\frac{3}{2}$	$k+1$	gravitino	$\frac{3}{2}k+\frac{3}{2}$	$\frac{k}{2}$
$\mathrm{Tr}(W^\alpha W_\alpha(\phi\psi)^k$	$\frac{3}{2}k+3$	$k+2$	vector	$\frac{3}{2}k+3$	$\frac{k}{2}$
$\mathrm{Tr}(J_{\alpha\dot{\alpha}}(\phi\psi)^k)$	$\frac{3}{2}k+3$	k	graviton	$\frac{3}{2}k+3$	$\frac{k}{2}$
$\mathrm{Tr}(e^V\bar{W}_{\dot{\alpha}}e^{-V}(\phi\psi)^k)$	$\frac{3}{2}k+\frac{3}{2}$	$k-1$	gravitino	$\frac{3}{2}k+\frac{3}{2}$	$\frac{k}{2}$
$\mathrm{Tr}(e^V\bar{W}_{\dot{\alpha}}e^{-V}W^2(\phi\psi)^k)$	$\frac{3}{2}k+\frac{9}{2}$	$k+1$	gravitino	$\frac{3}{2}k+\frac{9}{2}$	$\frac{k}{2}$

Here Δ is the conformal dimension, r is the R-charge and E_0 is the AdS energy. As an example, from the above identification we see that the operator $\mathrm{Tr}(\phi\psi)^k$ corresponds to vector multiplet containing scalars coming from the four form and graviton reduced on $\mathbb{T}^{1,1}$.

In our case, from the CFT side we need chiral dilaton multiplet of the type

$$\Phi^k = \mathrm{Tr}(W^\alpha W_\alpha(\phi\psi)^k)\,. \tag{4.1}$$

This has a conformal dimension $\frac{3}{2}k+3$ and R charge $k+2$ [39]. The trace is a symmetrized trace and indices are also symmetrized $\mathrm{SU}(2)\times\mathrm{SU}(2)$ indices. An important observation here is that the quartic superpotential $W = \epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}(\phi_i\psi_k\phi_j\psi_l)$ is not a chiral primary. Combined with another operator $\mathrm{Tr}(W^\alpha W_\alpha)$ — which is also not a chiral primary — this gives a chiral superfield which is $\bar{D}^2$ of the Konishi multiplet [39, 40, 41],

$$\bar{D}\bar{D}K = \bar{D}^2[\mathrm{Tr}(\phi e^V\bar{\phi}e^{-V}) + \mathrm{Tr}(\psi e^V\bar{\psi}e^{-V})]\,, \tag{4.2}$$

where $W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V$, V is a vector superfield and $\bar{D}$ is the operator which annihilates a chiral superfield S

$$\bar{D}_{\dot{\alpha}}S_{(\alpha_1\cdots\alpha_{2s_1})}(x,\theta,\bar{\theta}) = 0\,, \tag{4.3}$$

and $x, \theta, \bar{\theta}$ are the coordinate of a superspace. Therefore we have an equation of the form

$$\mathrm{Tr}(W^\alpha W_\alpha) \sim \bar{D}\bar{D}K - W\,. \tag{4.4}$$

The operator $\mathrm{Tr}(W^\alpha W_\alpha)$ is the one that appears in the supergravity spectrum and it coincides with the dilaton operator Φ^k with $k = 0$. Now we should ask what the dilaton operator corresponds to from the supergravity point of view. It contains a complex scalar coming from the two forms of type IIB on $\mathbb{T}^{1,1}$. This lies in the vector multiplet. The j, l values, which are the spin quantum numbers of the two SU(2), are given by [39, 40, 41]

$$j = l = \left|\frac{r-2}{2}\right| \equiv \frac{k}{2} = 0\,. \tag{4.5}$$

To calculate the mass of the state we define a quantity $H_0^- = H_0(j, l, r-2)$ where $H_0(j,l,r) = 6(j(j+1) + l(l+1) - \frac{r^2}{8})$. The mass of the state is given by

$$H_0^- + 1 \pm 2\sqrt{H_0^- + 4}\,. \tag{4.6}$$

Recall that this gives the AdS dual of a *combination* of the quartic superpotential and $\bar{D}\bar{D}K$. Therefore using this indirect method we can infer that a background of NSNS and RR two form is switched on. As observed in section 3 for the case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolded conifold with $\epsilon = -1$, the superpotential is a marginal deformation of the quartic superpotential of $\mathbb{T}^{1,1}$ [47]. Thus we expect the mass of the complex scalar state should be zero. By deforming the superpotential with $\bar{D}\bar{D}K$ kept fixed, we will be changing the background complex scalar field. This is how a marginal deformation of the superpotential can be related to NSNS and RR two forms.

At this point we can compare our result to the $\mathcal{N} = 4$ case. In $\mathcal{N} = 1$ language the Konishi multiplet is

$$K = \Phi_i e^V \bar{\Phi}_i\,, \tag{4.7}$$

where Φ_i, $i = 1, 2, 3$, are the chiral scalars. If we denote the cubic superpotential of $\mathcal{N} = 4$ as $W \sim \mathrm{Tr}([\Phi_1, \Phi_2]\Phi_3)$ then K satisfies [48]

$$\bar{D}\bar{D}K = W\,. \tag{4.8}$$

Identification of AdS dual is now simpler.

Another interesting example, though not directly related to our work, involving marginal deformation is given by the operator [42]

$$\mathcal{W} = \mathrm{Tr}(\phi_i\psi_j\phi_k\psi_l)(\sigma^r)^{ik}(\sigma^s)^{jl}\delta_{rs}\,. \tag{4.9}$$

This breaks the global symmetry from $\mathrm{SU}(2) \times \mathrm{SU}(2)$ to diagonal $\mathrm{SU}(2)$. The above identification of the dilaton multiplet and the scalar multiplet will give us the AdS dual of $\mathcal{W}$.

4.1 Supergravity duals and mirror symmetry

We study orbifolded conifold with discrete torsion in view of the AdS/CFT correspondence. In the large-N limit let us first investigate how orbifolded conifold encodes the discrete torsion. Note that the group $G = \mathbb{Z}_k \times \mathbb{Z}_k$ does not act freely on

$$\mathbb{T}^{1,1} \equiv \Big\{(x, y, u, v) \,|\, xy - uv = 0\,,\ |x|^2 + |y|^2 + |u|^2 + |v|^2 = 1\Big\}\,. \tag{4.10}$$

The first $\mathbb{Z}_k$ leaves a union of two linked circles

$$|x| = 1\,,\quad y = u = v = 0\,,\qquad |y| = 1\,,\quad x = u = v = 0\,, \tag{4.11}$$

fixed and the second $\mathbb{Z}_k$ leaves

$$|u| = 1\,,\quad x = y = v = 0\,,\qquad |v| = 1\,,\quad x = y = u = 0\,, \tag{4.12}$$

fixed. Along these cycles, we have locally a singularity of the type $\mathbb{C}^2/\mathbb{Z}_k$ where $\mathbb{Z}_k$ is the isotropy group at the point, that is, we will have a fibration of A_{k-1} singularities along this circle which resolves with an exceptional set of $(k-1)$ two spheres S_i as noticed in [46]. The intersection matrix of these spheres gives a Dynkin diagram of the A_{k-1} group. Of particular importance are the NSNS and RR two forms B_{NSNS} and B_{RR} which give rise to the scalars:

$$\xi_i^{\mathrm{NSNS}} = \int_{S_i} B_{\mathrm{NSNS}}\,,\qquad \zeta_I^{\mathrm{RR}} = \int_{S_i} B_{\mathrm{RR}}\,. \tag{4.13}$$

These fields are present even if we did not resolve the singularity in the string theory because they come from the twisted sector and they can survive in the supergravity limit if they correspond to massless particles. By going around the fixed circles (4.11) by the first $\mathbb{Z}_k$ on a closed loop, we are actually performing a twist by the elements of the second $\mathbb{Z}_k$ which don't fix the circle. For the twisted strings that live at the orbifold circles, going around the loop picks up a phase equal to the discrete torsion of the cycle acting on the group element to which the twisted state corresponds which is the discrete torsion phase in the partition function. In our case, it sets the boundary conditions for the massless sector states. Thus geometry differs from the standard $\mathbb{T}^{1,1}/G$ in that the singularities have monodromy of the exceptional spheres. We have the monodromies of the singularities which are located on a fixed circle and also the fractional B-field, this being characteristics of the conifold with discrete torsion.

Now we would like to consider a deformation of $\mathbb{T}^{1,1}$ corresponding to a marginal deformation in our superpotential. The arguments are essentially similar to those of [28]. It is observed in [35] that D3-branes in the presence of RR background fields picks up a dipole moment for higher brane charge and become extended in two additional dimensions. In our case, the AdS dual of a marginal deformation of the field theory is turning on RR and NS fields. In the presence of the RR field, the D3-branes become D5-branes wrapping two sphere which is the simplest possible configuration with the lowest energy. In other words, the RR background is given by the Hodge-dual $F_{(7)}$ which couples to a D5-brane, and is supported on $\mathbf{R}^{1,3} \times D^3$, where D^3 fills in S^2 and we can write $F_{(7)} = \tilde{F}_{(3)} \wedge dvol_4$. The 3-disc D^3 extends along the radial direction of AdS_5 whose boundary is a conformal compactification of $\mathbf{R}^{1,3}$. So we may write

$$F^{\rm RR}_{(3)} = d\rho \wedge \tilde{C}_{(2)} \,. \tag{4.14}$$

In this case the stretching happens mostly on the radial direction. But the Supergravity equations of motion imply that there is also a background $H^{\rm NS}_{(3)}$ field which does not have any component on the AdS directions (this is contrast with [43]–[46] where $H^{\rm NS}_{(3)}$ had a component on the radial direction and determined an RG flow in the field theory. Here we preserve the conformality). The field $H^{\rm NS}_{(3)}$ determines a stretching in the compact directions, therefore there is a combined effect of deformation, both in the radial and compact directions. The radius of the sphere is proportional with the flux of $\tilde{F}_{(3)}$ and $H^{\rm NS}_{(3)}$ through the 3-disc.

We now consider a large k branch where a D3-brane at the singularity becomes a set of k D5-branes wrapped on k S^2 cycles which in principle could be located at different location in the radial direction because of the dielectric effect due to the $H^{\rm RR}_{(3)}$ field. What happens if we bring all the k 2-cycle at the same radius and make any two of them meet at a single point? The intersection diagram of these k spheres is exactly the extended Dynkin diagram of the A_{k-1} group. The effect of $H^{\rm RR}_{(3)}$ disappears so that the dielectric effect disappears and only $H^{\rm NS}_{(3)}$ remains. As discussed in [28, 29], there are massless string modes stretching between the spherical D5-branes because the distance between neighboring spheres will become zero and these massless string modes will determine new branches in the moduli space which are not present for the case of non-zero RR field when the D5-branes are frozen. An explanation for this is that D5-branes on S^2 cycles are fractional branes which are known to be frozen at the singularity. In the case of aligned spherical D5-branes at the same radius, by turning on the massless modes at the intersections of the 2-spheres one resolves the pinched torus to a smooth $\mathbb{T}^2$ torus by giving sizes to the intersection points. In terms of the representations, the D5-brane wrapping different S^2 corresponds to different one-dimensional representations in (3.39). Since the sum of k irreducible representation on the singular point is a limit of an irreducible representation of a smooth point, we may move away from the singular point. Modification of the intersection points corresponds to move off a sum of fractional D5-branes from the singular points to form an integral D3-brane in the bulk. This new branch of the moduli signals two torus fibration of $\mathbb{T}^{1.1}$. The symmetry of two torus will give rise to the global $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry of the field theory of the orbifolded conifolds with discrete torsion. Thus the global $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry guides our geometric construction of two torus fibration of $\mathbb{T}^{1.1}$.

Before we give an explicit description we consider a Kähler deformation of the conifold $\mathcal{C} : xy - uv = 0$. We can make a Kähler deformation by means of blowing-up the singular point $(0, 0, 0, 0)$. This process will replace the singular point with a 4 manifold $\mathbb{P}^1 \times \mathbb{P}^1$. By rewriting eq. (3.1) as

$$y^2\left(\frac{x}{y} - \frac{u}{y}\frac{v}{y}\right) = v^2\left(\frac{x}{v}\frac{y}{v} - \frac{u}{v}\right) = 0\,, \tag{4.15}$$

we can see that the homogeneous coordinates of the first $\mathbb{P}^1$ (resp. the second $\mathbb{P}^1$) is given by $[x, y]$ (resp. $[u, v]$). Since $\mathbb{T}^{1,1}$ can be described as an intersection of the conifold (3.1) with a 7-sphere $|x|^2 + |y|^2 + |u|^2 + |v|^2 = 1$. it will be a $\mathrm{U}(1)$ bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.

Thus it is natural to consider two torus fibration determined by the arguments of $\phi_1\psi_1/\phi_2\psi_2$ and $\phi_1\psi_2/\phi_2\psi_1$. Here these correspond to the phases of the homogeneous coordinates of the two $\mathbb{P}^1$'s in the Kähler deformation above. The two phases now become S^1's in each one of the $\mathbb{P}^1$'s and they will give the two torus $\mathbb{T}^2$. Note that the global $\mathrm{U}(1)\times\mathrm{U}(1)$ acts on $\mathbb{T}^2$ freely. In terms of explicit coordinates, as discussed in [7], the $\mathbb{T}^{1,1}$ can be described by an $\mathrm{U}(1)$ fibration in the x^6 direction over $\mathbb{P}^1 \times \mathbb{P}^1$ whose basis lie in the $(4, 5, 8, 9)$ plane. The directions x^4, x^8 are taken to be the two S^1 cycles inside the two $\mathbb{P}^1$'s respectively and the $\mathbb{T}^2$ fibration of $\mathbb{T}^{1,1}$ will lie on the x^4, x^8 plane. This describes a torus fibration of the $\mathbb{T}^{1,1}$.

The AdS dual of the marginal deformation is given by turning on the RR and the NS fields. The NS field is turned on only in compact direction more precisely on the (x^4, x^5, x^6) or (x^6, x^8, x^9) directions and the RR field is turned on the (x^4, x^5, x^7) or (x^7, x^8, x^9) where x^7 is the radial direction.

The above discussion reminds the case of an orbifolded conifold without discrete torsion when a D3-brane orthogonal to the singularity was a D5-brane on a torus in the (x^4, x^8) directions and the brane configuration is a Brane box [49]. If we now deform to the degeneration, i.e. we approach a singular circle, the torus is split into k

spheres. In the case without discrete torsion they will correspond to a stripe of boxes or a stripe of diamonds in the x^4 or x^8 directions in a Brane box which was discussed in [46] to correspond to a fractional brane. In the discrete torsion case, a fractional brane will correspond to a D5-brane wrapped on each one of the k spheres. Therefore we have a correspondence between the fractional branes for the case without discrete torsion and with discrete torsion. This completes our discussion referring to the deformation of $\mathbb{T}^{1,1}$ five dimensional space into a $\mathbb{T}^2$ fibration.

What happens now if we make a T-duality with respect to the fibre $\mathbb{T}^2$? If $\mathbb{T}^2$ has k nodes and also wraps around n times before closing then it will describe an (n, k) doublet of charges which transforms under T-duality. At the singularities the Kähler form of the dual torus signals a B-field whose fractional part corresponds to the discrete torsion phase and we will also get the monodromies at the singularities as discussed at the beginning of this subsection. Therefore in the T-dual picture we have a D3-brane orthogonal to an orbifolded conifold with discrete torsion. So we have obtained that a T-duality on the $\mathbb{T}^2$ fibre takes us from the deformed $\mathbb{T}^{1,1}$ to an orbifolded conifold with discrete torsion.

5. Brane configuration on non-commutative torus

In this section we will compare the brane configuration from the orbifolded conifold with and without discrete torsion.

In the absence of discrete torsion the orbifolded conifold C_{kl} has a brane configuration in terms of intersecting NS5-branes to form a brane box. The total number of free parameters in this model come from $k + l - 2$ relative positions of the branes and kl intersections — which blow up to form a diamond [10]. This determines the cohomology $h^{1,1}$ as:

$$h^{1,1} = kl + k + l - 2 = (k + 1)(l + 1) - 3\,.$$

The branes configuration of the orbifolded conifold can be described on a torus. Once discrete torsion is introduced, the torus becomes non commutative. This would imply that the usual brane configurations are meaningless here. However we could also view the non-commutative moduli spaces as spaces with b units of background two form $B_{\rm NSNS}$. It turns out that with discrete torsion we also switch on a RR background of unit $c = 1 - \epsilon^{-1}$. Therefore the brane configuration is classified by (b, c) and we need sources for b, c, i.e. NS5-branes and Dp-branes, respectively.

A related configuration has recently been worked out in [50]. In their configuration the (b, c) values are realized by a configuration of orientifold planes, NS5-branes and Dp-branes. The orientifold plane is cut by both the types of branes. At the point where the NS5-branes cut the O-plane the b value jumps by N units by crossing the point where N is the number of NS5-branes. Similarly there is a shift of c when one crosses the Dp-branes. These b, c values specify the two discrete $\mathbb{Z}_2$ charges.

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When the NS5-brane is at the orientifold plane (O plane) it can split along the O-plane as two copies of $\frac{1}{2}$ NS5-branes [50]. In general the number n of $\frac{1}{2}$ NS5-branes is determined by

$$e^{i\int_{\mathrm{RP}^2} B_{\mathrm{NSNS}}} = (-)^n ,$$

where RP^2 is the space orthogonal to a O-plane.

Clearly our model should have similar kind of realization. Question is how do we realize the orientifold plane here? For this let us look at the $\mathbb{Z}_2$ actions of the conifold more carefully.

$$\mathbb{Z}_2 : (x, y, u, v) \longmapsto (x, y, -u, -v) , \qquad \mathbb{Z}_2 : (x, y, u, v) \longmapsto (-x, -y, u, v) . \tag{5.1}$$

Intersecting the above planes with the conifold $xy = uv$ we get a set of fixed lines

$$\begin{aligned} (x = u = v = 0) &\cup (y = u = v = 0) \\ \text{and } (x = y = u = 0) &\cup (x = y = v = 0) . \end{aligned} \tag{5.2}$$

These fixed lines are intersecting orbifold 5-planes away from the conifold point. At the conifold point they behave effectively as orbifold 3-planes.

Under an S-duality transformation the background metric of the system will not change and D3-brane will have strongly coupled gauge theory on its world volume. However the orbifold 5-plane will transform into a set of orientifold 5-plane and a D5-brane on top of each other. This is consistent with the fact that in general an orbifold plane supports gauge fields on its world volume. Under S-duality the combination O5-D5 will support gauge fields [51]. This is precisely how we can get orientifold planes in our model.

Observe that we could actually start from the near horizon geometry of the system. $AdS_5 \times \mathbb{T}^{1,1}/G$ will again have fixed orbifold 5-planes, which under S-duality become a system of O5-D5. As discussed in section 4, discrete torsion in the AdS limit is viewed as switching on background values of NSNS and RR three forms. This strongly suggest that we have to invoke sources for these forms in our model. Although the connection to [50] is suggestive here, there are still some points which need clarification. We will return to this in a future work.

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